VARIATIONAL PROBLEMS FOR SUPERSONIC BODIES OF REVOLUTION AND NOZZLES

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Variational problems in the gasdynamics of axisymmetric irrotational flows have been treated in a large number of papers up to the present time. The idea of considering a control contour, which appreciably simplifies the solution of the problems, was proposed by Nikol'skii in 1950 [1]. The method of solution of degenerate variational problems was worked out in 1946 by Okhotsimskii [2]. Guderley and Hantsche in 1955 [3] formulated the problem of the optimal supersonic nozzle and reduced it to a boundary problem for ordinary differential equations. In 1957 the author of the present paper published [4] the solution of a number of variational problems of gasdynamics of a perfect gas. The results of these papers, relating to axisymmetric nozzles, were repeated for the case of an imperfect gas by Rao in 1958 [5]. Rao's method differed from the method of [3,4], and its proof was given in 1959 by Guderley [6]. The cited papers touched on necessary conditions for an extremum, and [4] indicated a method of investigating the fulfillment of sufficient conditions. Fanselau in [7] returned to the study of sufficient conditions for an extremum, but did not obtain constructive results. Finally, Sternin in 1961 [8] derived the equation of the locus of points of extremal characteristics, at which the acceleration became infinite, and at the same time determined the region of applicability of the previously worked out solution with continuous functions. Here is developed a further study of variational problems for axisymmetric supersonic flows. Sufficient conditions are determined for attaining the minimum wave drag of bodies of revolution, discontinuous solutions are constructed for regions in which the minimum is not attained with continuous functions, and regions of isentropic flow are delineated.

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computations, and to L.V. Papandin for executing the graphs.

1. The variational problem. Axisymmetric flows of a perfect gas satisfy the equations

$$\frac{\partial r\rho w\cos\vartheta}{\partial x}+\frac{\partial r\rho w\sin\vartheta}{\partial r}=0$$

$$w\cos\vartheta\,\frac{\partial w\cos\vartheta}{\partial x}+w\sin\vartheta\,\frac{\partial w\cos\vartheta}{\partial r}+\frac{1}{\rho}\,\frac{\partial p}{\partial x}=0 \qquad (1.1)$$

$$\frac{w^2}{2}+\frac{\varkappa}{\varkappa-1}\,\frac{p}{\rho}=\frac{1}{2}\,\frac{\varkappa+1}{\varkappa-1}$$

$$\frac{p}{\rho^{\varkappa}}=\begin{cases}\text{const} - & \text{in the irrotational case}\\ \varphi^{\circ}(\psi) - & \text{in the rotational case}\end{cases}$$

Here x, r denote Cartesian coordinates in a meridional plane of the flow; w is the velocity divided by the critical velocity of sound a_*; ϑ is the angle of inclination of the velocity to the axis of the stream x; ρ is the density of the gas, divided by the stagnation density ρ_0; p is the pressure, divided by the product $\rho_0 a_*^2$; κ is the adiabatic index; ψ is the stream function

$$(d\psi=r\rho w\,(\cos\,\vartheta dr-\sin\,\vartheta dx)).$$

On the characteristics of the system (1.1) the following equations are satisfied:

first kind:

$$dr=\tan(\vartheta+\alpha)\,dx$$

$$d\vartheta+\frac{1+\cos 2\alpha}{\varkappa-\cos 2\alpha}\,d\alpha+\frac{\sin\vartheta\sin\alpha}{\sin(\vartheta+\alpha)}\,\frac{dr}{r}-\frac{\sin 2\alpha}{2\varkappa(\varkappa-1)}\,d\ln\varphi^{\circ}=0 \qquad (1.2)$$

second kind:

$$dr=\tan(\vartheta-\alpha)\,dx$$

$$d\vartheta-\frac{1+\cos 2\alpha}{\varkappa-\cos 2\alpha}\,d\alpha-\frac{\sin\vartheta\sin\alpha}{\sin(\vartheta-\alpha)}\,\frac{dr}{r}+\frac{\sin 2\alpha}{2\varkappa(\varkappa-1)}\,d\ln\varphi^{\circ}=0 \qquad (1.3)$$

where a is the Mach angle, related to w by the equation $w^2(\kappa-\cos 2a)=\kappa+1$.

Suppose that points a and b are given in the xr plane, together with a characteristic of the first kind ae, and moreover the entropy function ϕ° on ae is constant (Fig. 1). It is required to construct the profile ab which secures the minimum wave drag. The problem is solved by determining the corresponding characteristic of the second kind bc passing through the point b.

Let us introduce the following notation: P, A, Θ are r, α, ϑ respectively on the characteristic ae

$$\sigma(\alpha) = \left(\frac{\varkappa - 1}{\varkappa - \cos 2\alpha}\right)^{\frac{1}{2}}, \qquad \tau(\alpha) = \left(\frac{1 - \cos 2\alpha}{\varkappa - \cos 2\alpha}\right)^{\frac{1}{2}\frac{\varkappa+1}{\varkappa-1}}$$

$$F_1 = \int_{P=P_a}^{r_c} \sigma(A)\,\tau(A)\left[\frac{\sin A}{\varkappa} + \frac{\cos\Theta}{\sin(\Theta + A)}\right] P\,dP$$

$$F_2 = \int_{P=P_a}^{r_c} \cot(\Theta + A)\,dP, \qquad F_3 = \int_{P=P_a}^{r_c} \frac{\tau(A)\,P\,dP}{\sin(\Theta + A)}$$

$$\Phi_1 = \sigma(\alpha)\,\tau(\alpha)\left[\frac{\sin\alpha}{\varkappa} - \frac{\cos\vartheta}{\sin(\vartheta - \alpha)}\right] r\varphi^{-\frac{1}{\varkappa-1}}$$

$$\Phi_2 = \cot(\vartheta - \alpha), \qquad \Phi_3 = \frac{\tau(\alpha)\,r}{\sin(\vartheta - \alpha)}\,\varphi^{-\frac{1}{\varkappa-1}}, \qquad \varphi = \exp\frac{s(r) - S}{c_v}$$

Here $s(r)$ is the entropy of the gas on bc, S is the entropy on ae, c_v is the specific heat of the gas at constant volume; the function ϕ is the ratio of ϕ° on bc to the constant ϕ° on ae.

First of all we shall assume that the flow in the triangle abc is irrotational, i.e. $s(r) \equiv S = \text{const}$ or $\phi = 1$. In the sequel this assumption will receive special attention.

For the characteristic bc the following variational problem arises [4].

For given constants r_a, r_b, $X = x_b - x_a$ and given functions $A(P)$, $\Theta(P)$, to find the function $\alpha(r)$ realizing the extremum of the functional

$$\chi = F_1(r_c) - \int_{r=r_b}^{r_c} \Phi_1(r, \alpha, \beta)\,dr \tag{1.4}$$

under the isoperimetric conditions

$$X = F_2(r_c) - \int_{r=r_b}^{r_c} \Phi_2(\alpha, \beta)\,dr, \qquad \Psi \equiv 0 = F_3(r_c) + \int_{r=r_b}^{r_c} \Phi_3(r, \alpha, \beta)\,dr \tag{1.5}$$

and the differential relation

$$\Phi_4 \equiv \frac{d\beta}{dr} - \frac{\sin[\beta - f(\alpha)]\sin\alpha}{r\sin[\beta - f(\alpha) - \alpha]} = 0 \tag{1.6}$$

The quantity χ determines to within a constant factor the wave resistance of the body of revolution, X is the length of the projection of the profile ab on the x-axis, $\Psi = 0$ is the [nett] flux of gas across the characteristic contour acb. The function β is connected with ϑ and a by the equations

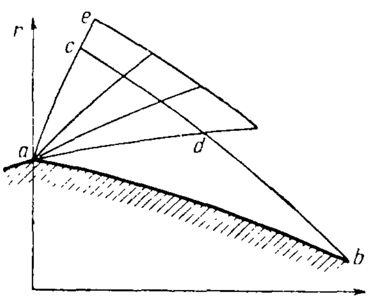

Fig. 1.

$$\beta = \vartheta + f(\alpha) \tag{1.7}$$

$$f(\alpha) = \sqrt{\frac{\varkappa+1}{\varkappa-1}} \tan^{-1}\left(\sqrt{\frac{\varkappa-1}{\varkappa+1}} \cot \alpha\right) + \alpha$$

The condition (1.6) coincides with the second equation of (1.3). Here and in the sequel the calculations are carried through in the system of variables r, a, β, but for purposes of presentation the formulas are written in terms of the variables r, a, ϑ.

Admissible functions $a(r)$ must satisfy the following requirements, determined by the properties of solutions of Equations (1.1) in problems of flow past a surface. The functions $a(r)$ are piecewise continuous. In the intervals in which $a(r)$ is continuous, the first derivative of $a(r)$ must not exceed certain limits, set by the bend in the contour bounding the stream [9]. The conditions at the points of discontinuity of $a(r)$ must be given special consideration, moreover discontinuities at shock waves are not considered because of the assumption as to constancy of entropy.

2. Isentropic discontinuities. A discontinuity of the functions on a characteristic of the second kind, accompanied by conservation of entropy, is possible in the case of focusing of characteristics of the first kind at a point d, situated on the characteristic under consideration (Fig. 2). For the relations obtained at the point d it is sufficient to consider plane flow with straight line characteristics. Suppose that at the point d the pencil of characteristics adk is focused. The intersection of the characteristics results in the formation of the shock wave dn. The reflection of the disturbance occurs either as the pencil of characteristics ldg, or else as a shock wave. We shall study only the first case. The line df represents the contact discontinuity.

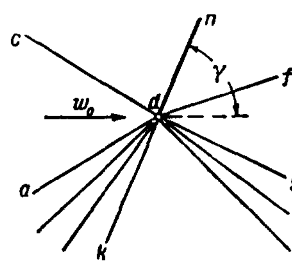

Fig. 2.

The quantities a, ϑ, p, constants in the regions nda, kdl, gdf and *fdn*, will be distinguished by the indices 0, 1, 2 and 3, respectively.

The Prandtl-Meyer solution gives

$$\vartheta_0 + f(\alpha_0) = \vartheta_1 + f(\alpha_1), \qquad \vartheta_1 - f(\alpha_1) = \vartheta_2 - f(\alpha_2) \tag{2.1}$$

where the function f is defined in (1.7). Moreover,

$$\alpha_0 \leqslant \alpha_1, \qquad \alpha_1 \geqslant \alpha_2 \tag{2.2}$$

The angle of inclination of the velocity ϑ_3 and the pressure p_3, divided by the product of the gas density in the region *nda* and the square of the critical speed, are determined by the following relations at the shock wave

$$\begin{aligned} \vartheta_3 &= \vartheta_0 + \gamma - \tan^{-1} \frac{\varkappa - \cos 2\alpha_0 - (\varkappa - 1)\cos^2\gamma}{(\varkappa + 1)\sin\gamma\cos\gamma} \\ p_3 &= \frac{2\sin^2\gamma}{\varkappa - \cos 2\alpha_0} - \frac{\varkappa - 1}{2\varkappa}\,\frac{1 - \cos 2\alpha_0}{\varkappa - \cos 2\alpha_0} \end{aligned} \tag{2.3}$$

where γ is the angle of inclination of the shock wave *dn* to the direction of the velocity w_0.

The pressure p_2 is equal to

$$p_2 = \frac{\varkappa + 1}{2\varkappa}\left(\frac{1 - \cos 2\alpha_0}{\varkappa - \cos 2\alpha_0}\right)^{-\frac{1}{\varkappa - 1}}\left(\frac{1 - \cos 2\alpha_2}{\varkappa - \cos 2\alpha_2}\right)^{\frac{\varkappa}{\varkappa - 1}} \tag{2.4}$$

On the line df of the contact discontinuity the following relations hold

$$\vartheta_2 = \vartheta_3, \qquad p_2 = p_3 \tag{2.5}$$

Accordingly, the ten quantities α_0, ϑ_0, α_1, ϑ_1, α_2, ϑ_2, p_2, ϑ_3, p_3, γ are connected by the seven equations (2.1), (2.3) to (2.5). For example, if the quantities α_0, ϑ_0 α_1 are given, they determine the remaining quantities. The parameter ϑ_0 is unnecessary, since its variation merely rotates the whole flow pattern.

In accordance with what has been said in this section, we shall assume that the function $\alpha(r)$ is admissible if

$$\alpha_1 \leqslant \pi / 2 \tag{2.6}$$

and if at the point of discontinuity d of the characteristic bc the quantity α undergoes a jump from α_0 to a certain α_4 satisfying the inequalities

$$\alpha_2 \leqslant \alpha_4 \leqslant \alpha_1 \tag{2.7}$$

where α_2 is related to α_0, ϑ_0, and α_1 by the equations (2.1), (2.3) to (2.5) and the inequalities (2.2).

The results of computations of these equations with $\kappa = 1.4$ are presented in Fig. 3. Suppose that for a given α_0 and a certain angle of the shock wave $\gamma = \gamma_*(\alpha_0)$ the velocity of sound is attained behind the shock wave. Then for the various combinations of α_0 and α_1 we determine the corresponding value of the parameter h

$$h = \frac{\gamma - \alpha_0}{\gamma_* - \alpha_0}$$

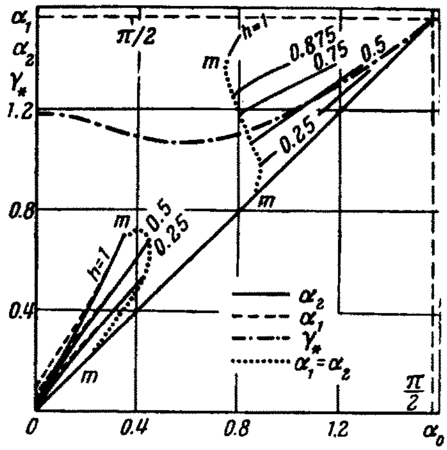

Fig. 3.

Fig. 3 shows the dependence of α_2 on the values of h and α_0. The value of α_1 is shown only where it is appreciably greater than α_2. In the same place we display the function $\gamma_*(\alpha_0)$. The dotted curves mm correspond to $\alpha_1(\alpha_0, h) = \alpha_2(\alpha_0, h)$ and delimit the regions of existence of solutions of the form under consideration. The regions not enclosed by these curves correspond to the occurrence of a reflected shock wave.

3. The extremal field. The variational problem of Section 1 is reduced by the method of Lagrange's multipliers to the problem of the unconditional extremum for the functional

$$I = F(r_c) + \int_{r=r_b}^{r_c} \Phi(r, \alpha, \beta, \beta', \nu)\, dr \tag{3.1}$$

where

$$F = F_1 + \lambda F_2 - \mu F_3, \qquad \Phi = -[\Phi_1 + \lambda\Phi_2 + \mu\Phi_3 + \nu\Phi_4]$$

λ and μ are constants, whilst ν is the variable Lagrange multiplier.

In the case of continuous functions $\alpha(r)$ this problem is solved in [4]. The results which will be necessary later may be reduced to the following.

If the variations are carried out in accordance with the restrictions of the problem, the conditions of transversality are fulfilled, and ν is chosen so that the variation with respect to β is zero, then the first variation

$$\delta\chi = \int_{r=r_b}^{r_c} \Phi_\alpha \,\delta\alpha dr \tag{3.2}$$

where

$$\Phi_\alpha = \frac{1}{(\varkappa - \cos 2\alpha)\sin^2(\vartheta - \alpha)} \Big\{\sigma(\alpha)\,\tau(\alpha)\Big[\frac{\varkappa+1}{2}\frac{\sin 2\vartheta}{\sin\alpha} - 2\cos\alpha\,(1+\cos 2\alpha)\Big] r + \lambda\,(1 - \varkappa + 2\cos 2\alpha) + \mu\tau(\alpha)\,[(1-\varkappa+2\cos 2\alpha)\cos(\vartheta-\alpha) - (\varkappa+1)\cot\alpha\sin(\vartheta-\alpha)]\, r - \frac{\nu}{2r}[\sin^2 2\alpha - 2(\varkappa - \cos 2\alpha)\sin^2\vartheta]\Big\} \tag{3.3}$$

The value of ν is determined from the equation

$$\Phi_\beta - \frac{d\nu}{dr} = 0 \tag{3.4}$$

where

$$\Phi_\beta = -\frac{1}{\sin^2(\vartheta-\alpha)}\Big\{r\tau(\alpha)\,[\sigma(\alpha)\cos\alpha - \mu\cos(\vartheta-\alpha)] - \lambda + \frac{\nu}{r}\sin^2\alpha\Big\} \tag{3.5}$$

Moreover, the required characteristic *bc* consists of two segments *cd* and *db* (Fig. 1). The segment *cd* is determined by the characteristic *ac* and the bend in the profile *ab* at the point *a*. The derivative of $a(r)$ along *cd* assumes the limiting value [9]. On the segment *db* the functions satisfy Euler's equations for the given problem $\Phi_\alpha = 0$ and (3.4) and, taking into account the condition of transversality, are determined by the equations

$$\nu \equiv 0, \qquad r\tau(\alpha)\,\sigma(\alpha)\frac{\sin^2\vartheta}{\cos\alpha} = \lambda, \qquad \sigma(\alpha)\frac{\cos(\vartheta+\alpha)}{\cos\alpha} = \mu \tag{3.6}$$

The values of λ and μ are found from the latter equations with r, a, ϑ taken at the point d. The Equations (3.6) can be solved with respect to r and ϑ

$$r\frac{\tau(\alpha)}{\sigma(\alpha)}[\sigma^2(\alpha)\cos\alpha - \mu^2\cos\alpha\cos 2\alpha - \operatorname{sign}(\vartheta_d + \alpha_d)\,\mu\sin 2\alpha\sqrt{\sigma^2(\alpha) - \mu^2\cos^2\alpha}\,] = \lambda \tag{3.7}$$

$$\vartheta = \operatorname{sign}\vartheta_d \left|\sin^{-1}\left[\sqrt{\frac{\lambda\cos\alpha}{r}}\left(\frac{1-\cos 2\alpha}{\varkappa - \cos 2\alpha}\right)^{-\frac{1}{4}\frac{\varkappa+1}{\varkappa-1}}\left(\frac{\varkappa - \cos 2\alpha}{\varkappa+1}\right)^{\frac{1}{4}}\right]\right| \tag{3.8}$$

Finally, we notice that by virtue of the second equation (3.6) with $\alpha \neq \pi/2$ the value of ϑ cannot change sign, and $\lambda \geqslant 0$.

Equation (3.7) makes it possible to construct the function $\alpha(r/\lambda, \mu)$. In Fig. 4 are depicted the curves $\alpha(R)$ for different values of μ in the case $\kappa = 1.4$. The value of R is determined by the equation

$$R = \frac{r}{10\lambda + r} \qquad (3.9)$$

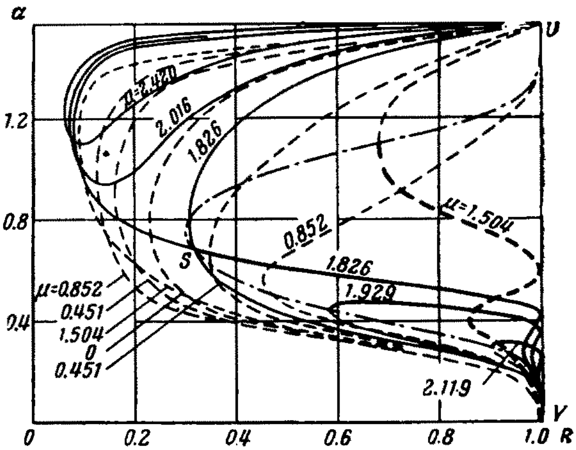

Fig. 4.

Equation (3.8) or the third equation of (3.6) gives the function $\vartheta(\alpha, \mu)$. The corresponding curves for $\kappa = 1.4$ are portrayed in Fig. 5. The extremals in the velocity hodograph plane are drawn in Fig. 6.

The greatest value of $\mu = \mu_*$ for which the radical in (3.7) is real in the whole interval $0 \leqslant \alpha \leqslant \pi/2$, is

$$\mu_* = \sqrt{\frac{8}{\varkappa + 1}}$$

Suppose that the values of $\alpha = \alpha_*$, $\vartheta = \vartheta_*$ are determined by the equations

$$2 \cos 2\alpha_* = \varkappa - 1, \cos(\vartheta_* + \alpha_*) = 1.$$

The point with these values of μ and α is a saddle-point in the (α, ϑ) plane and is indicated in Figs. 4-6 by the letter S. Through the point S pass two extremals with $\mu = \mu_*$. The point U with $\alpha = \pi/2$, $\vartheta = 0$ is a focal point.

The extremals in the $R\alpha$ plane with $\mu > \mu_*$ are loops. When $\mu < \mu_*$ the extremals join the points $R = 1$, $\alpha = 0$ and $R = 1$, $\alpha = \pi/2$. The correspondence of the regions in the $R\alpha$, $\alpha\,\vartheta$ and $w\,\vartheta$ planes is easily followed using the values of μ shown on Figs. 4-6.

4. The region of minimal resistance. Along the characteristics of the second kind $dr = -\sin(\vartheta - \alpha)dl$, where l is the arc length of the characteristic, measured from the point b and directed towards increasing gas discharge ψ, i.e. in the direction of the point c.

Using the expression for dr, let us rewrite (3.2) in the form

$$\delta\chi = -\int_{l=0}^{l_c} V\delta\alpha dl \qquad (V = \Phi_\alpha \sin(\vartheta - \alpha)) \tag{4.1}$$

We notice that the signs of dr and dR coincide by virtue of Equation (3.9) and the inequality $\lambda > 0$.

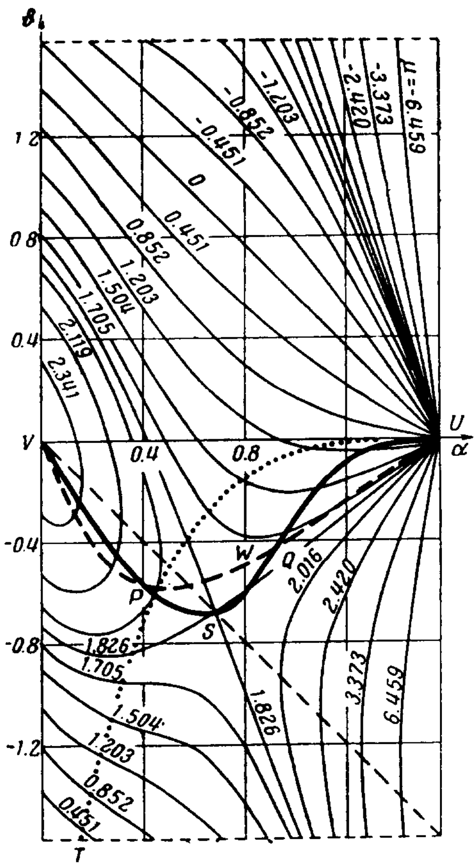

Fig. 5.

Let us consider now any extremal (Fig. 7) in the $R\alpha$ plane. On it $V = 0$ and the first variation $\delta\chi$ vanishes in accordance with (4.1). A certain segment of the extremal, on which $\alpha(R)$ is a single-valued function, secures minimum wave resistance χ if the inequality $V < 0$ is valid above it ($\delta\alpha > 0$) in the $R\alpha$ plane, whilst below it ($\delta\alpha < 0$) the inequality $V > 0$ holds. In fact, the expression (4.1) shows that in this case $\delta\chi > 0$ whatever the sign of $\delta\alpha$. Accordingly the problem consists of determining the sign of V in the neighborhood of the given segment of the extremal in the $R\alpha$ plane.

From the character of the extremal in Equation (4.1) it follows that a segment of the extremal defining minimum χ can be joined by segments giving maximum χ, at points where the derivative $d\alpha/dr$ is infinite with $\sin(\vartheta - \alpha) \neq 0$. These conditions are fulfilled at points with infinite accelerations (or $d\alpha/dl$). Sternin [8] found the geometric locus of points with infinite derivative $d\alpha/dr$ by differentiating the expression (3.7) with respect to α, eliminating the quantity μ by means of (3.6) and equating to zero the resulting expression. In the case under consideration with study of extremal characteristics of the second kind, the equation of the geometric locus of points with $dr/d\alpha = 0$ obtained in [8] has the form

$$T(\alpha, \vartheta) \equiv (\varkappa - 1 + 2\cos^2 2\alpha)\cos\alpha - (\varkappa + 1)\cos\alpha\cos 2\vartheta - (\varkappa - 1 - 2\cos 2\alpha)\sin\alpha\sin 2\vartheta = 0 \tag{4.2}$$

This equation has the simple root $\vartheta = \alpha + n\pi$, where n is an integer, corresponding to a finite acceleration, and also the roots

$$\vartheta = -\tan^{-1}\frac{(1+\cos 2\alpha)\sin 2\alpha}{\varkappa + \cos^2 2\alpha} + n\pi \tag{4.3}$$

indicated in Fig. 5 for $n = 0$ by the curve VSU. It is easy to show that this curve passes through the point S. Here and later the roots of Equation (4.2) with $n \neq 0$ will not receive special consideration, as their meaning is obvious.

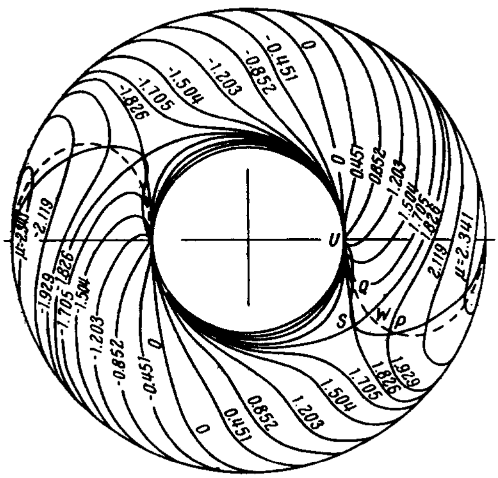

Fig. 6.

The curve defined by Equation (4.3) in the $R\alpha$ plane intersects once the loop-shaped extremals with $\mu > \mu_*$ and $\alpha \leqslant \alpha_*$, twice the loop-shaped extremals with $\mu > \mu_*$ and $\alpha > \alpha_*$, and again once it intersects the part of the extremals with $\mu < \mu_*$.

Let us turn to the determination of the sign of V outside the extremal under consideration and to finding the meaning of the function (4.3) from the point of view of sufficient conditions for an extremum. The quantity V is a functional and depends on the path $\alpha = \alpha(R)$, joining a certain point h of the extremal with a point g under consideration (Fig. 7). In fact, the quantity ϑ is connected with α and r by Equations (1.6) and (1.7), whilst ϑ_g depends at the same time on ϑ_h, α_h and $\alpha(r)$ on hg. Singular points V on the extremals are points at which $\sin(\vartheta - \alpha) = 0$, as follows from (4.1) and (3.3). It is essential, however, that V preserves its sign in such neighborhoods of regular points of extremals which do not intersect the extremal under consideration. In the opposite case the neighborhoods of the extremal being considered intersect new extremals $\Phi_\alpha = 0$, but outside the curve $\Phi_\alpha = 0$ already known. Accordingly, it is sufficient to determine by any one path the sign of V outside the extremals. We shall

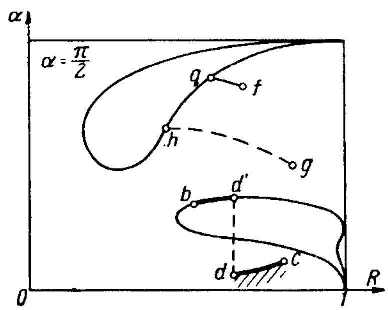

Fig. 7.

choose the following path. In the neighborhood of a regular point q let us construct an infinitely small element of the characteristic qf not coinciding with the extremal. The quantities $\delta\alpha$ and dr on the element qf have an order of smallness. The quantity V at the point f, with accuracy to a quantity of the first order of smallness, is given by

$$V = \delta V = V_\alpha \delta\alpha + V_\beta \delta\beta + V_\nu \delta\nu \tag{4.4}$$

Let us calculate the variations $\delta\beta$ and $\delta\nu$. Varying the Equation (1.6) and carrying out the integration, we obtain

$$\delta\beta = E(r_f) \int\limits_{r=r_q}^{r_f} \frac{B}{E(r)}\, \delta\alpha\, dr \tag{4.5}$$

Here

$$E(r) = \exp \int\limits_{r=r_q}^{r} \frac{\sin^2\alpha}{r\sin^2(\vartheta-\alpha)}\, dr$$

$$B = \frac{\sin\vartheta \sin\alpha}{r(\varkappa - \cos 2\alpha)\sin(\vartheta-\alpha)} [(1+\cos 2\alpha)\cot\vartheta +$$
$$+ (\varkappa - \cos 2\alpha)\cot\alpha - (2\cos 2\alpha - \varkappa + 1)\cot(\vartheta - \alpha)]$$

The quantity $\delta\nu$ at the point f is equal to ν, since $\nu_q = 0$, and it is determined by Equations (3.4) and (3.5). Moreover $\Phi_3 = 0$ at the point q. Therefore on the element qf we have, with accuracy to a quantity of the first order of smallness

$$\Phi_\beta = (\Phi_\beta|_{\nu=0})_\alpha\, \delta\alpha + (\Phi_\beta|_{\nu=0})_\beta\, \delta\beta + \Phi_{\beta\nu}\delta\nu$$

Integrating (3.4) and allowing for the last equation gives

$$\delta\nu = E(r_f) \int\limits_{r=r_q}^{r_f} \frac{(\Phi_\beta|_{\nu=0})_\alpha\delta\alpha + (\Phi_\beta|_{\nu=0})_\beta\delta\beta}{E(r)}\, dr \tag{4.6}$$

Equations (4.5) and (4.6) show that the variations $\delta\beta$ and $\delta\nu$ on the infinitely small element qf have by comparison with $\delta\alpha$ a higher order of smallness. Hence also from (4.4) it follows that, with accuracy to a quantity of the first order of smallness

$$\delta V = V_\alpha \delta\alpha \tag{4.7}$$

Let us recall Expression (4.1) for V and (3.3) for Φ_α. Let us differentiate V with respect to α, as always in the system of variables r, α, β, and let us substitute for λ and μ from (3.6) in the resulting expression. Eventually we obtain when $V = 0$

$$V_\alpha = -\frac{2r\tau(\alpha)\sigma(\alpha)\sin 2\alpha}{(\varkappa - \cos 2\alpha)^2(1 - \cos 2\alpha)} \frac{T(\alpha, \vartheta)}{\sin(\vartheta - \alpha)} \tag{4.8}$$

where $T(\alpha, \vartheta)$ is defined in (4.2). The roots of the equation $T(\alpha, \vartheta) = 0$ have already been found. From Formula (4.8) it follows that $V_\alpha = 0$ with ϑ determined by the Equations (4.3). It is not difficult to see that $V_\alpha < 0$ when

$$H(\alpha) < \vartheta < \pi + H(\alpha) \tag{4.9}$$

where

$$H(\alpha) \equiv - \tan^{-1} \frac{(1 + \cos 2\alpha)\sin 2\alpha}{\varkappa + \cos^2 2\alpha}$$

i.e. in the region above the curve VSU (Fig. 5), and $V_\alpha > 0$ when

$$-\pi + H(\alpha) < \vartheta < H(\alpha) \tag{4.10}$$

i.e. in the region below the curve VSU.

In the region (4.9) the sign of V in the neighborhood of the extremal is opposite to the sign of $\delta\alpha$ by virtue of (4.7). Hence also from (4.1) we conclude that the region (4.9) corresponds to minimum drag χ.

Similarly it can be established that the region (4.10) corresponds to maximum χ.

Let us return to the $R\alpha$ plane (Fig. 4). Here the minimum drag corresponds to the segment of the extremals cut off by the chain-dotted curve VSU and depicted by the heavier curves.

The connection between the results of [8] and those obtained here is established without difficulty. In fact, the derivative $dr/d\alpha$ along the extremal, calculated by means of the equations

$$\Phi_\alpha(r, \alpha, \beta)|_{v=0} = 0, \quad \Phi_\beta(r, \alpha, \beta)|_{v=0} = 0$$

fulfilled on the extremals and considered as an implicit expression for r in terms of α, is given by

$$\frac{dr}{d\alpha} = \frac{\Phi_{\alpha\alpha}\Phi_{\beta\beta} - \Phi_{\alpha\beta}^2}{\Phi_{\beta r}\Phi_{\alpha\beta} - \Phi_{\alpha r}\Phi_{\beta\beta}} \tag{4.11}$$

On the extremals we have from (4.8)

$$\Phi_{\alpha\alpha} = -\frac{2r\tau(\alpha)\sigma(\alpha)\sin 2\alpha T(\alpha, \vartheta)}{(\varkappa - \cos 2\alpha)^2(1 - \cos 2\alpha)\sin^2(\vartheta - \alpha)} \tag{4.12}$$

Similarly we can calculate (4.13)

$$\Phi_{\alpha\beta} = -\frac{r\tau(\alpha)\sigma(\alpha)\sin 2\alpha T(\alpha, \vartheta)}{(\varkappa - \cos 2\alpha)(1 - \cos^2 2\alpha)\sin^2(\vartheta - \alpha)}, \quad \Phi_{\beta\beta} = -\frac{r\tau(\alpha)\sigma(\alpha)\cos(\vartheta + \alpha)}{\cos\alpha\sin(\vartheta - \alpha)}$$

Accordingly, the numerators of the first and second terms [sic] on the right-hand side of (4.11) vanish with ϑ, defined by Expression (4.3), whilst the denominator does not vanish then.

5. Discontinuous solutions. By virtue of the degeneracy of the variational problem under consideration, a two-sided extremum is not, generally speaking, attainable therein. A boundary extremum can be attained on the limiting permissible functions a or r, discussed in Sections 1 and 2. The limiting decrease of a on the characteristic in moving from the point c to the point b is determined [10,11] by the bend in the profile ab at the point a. The limiting increase of a can be achieved with the discontinuous functions considered in Section 2.

The characteristics of a flow of the first type in the region cad are depicted in Fig. 8. In this flow pattern let us draw the curve at, on which a and ϑ are connected by Equation (4.3) with $n = 0$. The points of the region cat satisfy the conditions (4.9). In this case from each point of the region cat we can start an extremal characteristic, defining a body with minimal drag. Such continuous solutions were obtained in [4], the region of their existence is bounded by the appearance of the extremal on the curve VSU (Fig. 5), i.e. by the occurrence of points with infinite accelerations.

The set of points of the region cat, or at least a part of it, corresponds to the set of points b, which can be assumed given by the formulation of the problem. The class of solutions of the variational problem is extended if we succeed in finding the profile ab with minimal drag corresponding in this sense to the points d from (4.10).

Let us find the solution of the problem for the case when the starting point d of the extremal (Fig. 8) belongs to the region (4.10). After constructing a scheme for the solution it is necessary to prove the fulfillment of the necessary and sufficient conditions for a minimum χ for all the solutions as a whole.

Accordingly, the point d belongs to the region (4.10). It is evident that from this point it is necessary to pass by a certain path to the region (4.9). For each permissible continuous transition, a part of the characteristic will belong to the region (4.10) and therefore it can be varied in such a way that the quantity χ decreases. It remains to make use of discontinuous transitions from one region to the other. According to the conditions of the problem we allow only isentropic discontinuities, caused by focusing of the characteristics of the first kind adk at the point d (Fig. 8). Such a transition in the $a\vartheta$ plane is traced out (Fig. 9) by the curve d_0d_1, satisfying the equation of the characteristics of the second kind (1.3) with $dr = d\phi^\circ = 0$

$$\vartheta_0 + f(\alpha_0) = \vartheta_1 + f(\alpha_1)$$

and by the curve $d_1 d_4$, satisfying the equation of characteristics of the first kind (1.2) with $dr = d\phi^\circ = 0$

$$\vartheta_1 - f(\alpha_1) = \vartheta_4 - f(\alpha_4)$$

The new flow pattern retains supplementary degrees of freedom connected with the position of the point d (Fig. 8). Let us introduce the conditions of transversality corresponding to these degrees of freedom.

The functional I, expressed by Formula (3.1), can be written in the equivalent form [4]

$$I = F(r_c) - \int_{r=r_c}^{r_d} \Phi dr + \int_{r=r_b}^{r_d} \Phi dr \tag{5.1}$$

where

$$F(r_c) - \int_{r=r_c}^{r_d} \Phi dr = F(r_d) \tag{5.2}$$

Here $F(r_d)$ is the same integral as $F(r_c)$ in (3.1), but taken along the characteristic ad. When calculating the first variation it is necessary to allow for the structure of the flow in the neighborhood of the point d, described in Section 2. The values at the point d obtained as one approaches this point along the characteristics ad, kd and bd, are denoted respectively by the indices 0, 1 and 4.

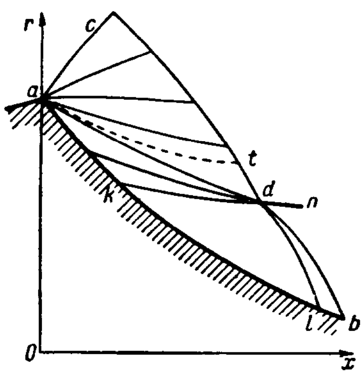

Fig. 8.

The flow in cad is fixed; only the choice of the boundaries ad and dc of this region is arbitrary. The angle ω formed by the tangent to the characteristics ad and kd at the point d is not fixed and can be varied. On the segment db the function $a(r)$ is free, but belongs to the class of permissible function of the variational problem. Accordingly, between the end values of the problem the two coordinates for the point d are arbitrary, completely determining the region cad, the quantity ω and the terminal value r_d of the segment of the characteristic db, which upon variation can move along the fixed characteristic kd. The character of the latter variation is determined by the fact that by virtue of the independence of the solution on *ak* from the segment *kb* this latter segment must have minimal wave resistance.

As the directions in which to vary the location of the point *d*, let us take the tangents at the point *d* to the characteristics *cd*, *ad* and *kd*, and the corresponding variations in the quantity *r* will be denoted by δr_{dc}, δr_{da} and δr_{dk}.

With the variation δr_{da} it makes the transition from the characteristic *cd* to another characteristic of the second kind in the same region *cad*. Taking account of Equation (5.2), the increment in the first two terms of (5.1) can be written as

$$[dF(r_0) / dr_0]\,\delta r_{da}$$

whilst the increment in the third term on account of the variation δr_{da} is $\Phi_4 \delta r_{da}$. A similar form for the fixed characteristic *kd* is calculated for the variation depending on δr_{dk}. The part δI_{dak} of the variation of the functional *I*, depending on the variations δr_{da} and δr_{dk}, takes the form

$$\delta I_{dak} = \left(\frac{dF(r_0)}{dr_0} + \Phi_0\right)\delta r_{da} + \left(\frac{dF(r_4)}{dr_4} + \Phi_4\right)\delta r_{dk} \tag{5.3}$$

The variation in the direction of the characteristic of the second kind can be carried out for *I* in the form (5.1). Here the quantity r_d changes by δr_{dc}, the second term in (5.1) decreases by $\Phi_0 \delta r_{dc}$, whilst the third increases by $\Phi_4 \delta r_{dc}$. Accordingly, the part δI_{dc} of the variation of the functional *I*, depending upon δr_{dc}, is equal to

$$\delta I_{dc} = (\Phi_4 - \Phi_0)\,\delta r_{dc} \tag{5.4}$$

The first variation, depending on $\delta\omega$ vanishes since only the end value of Φ_4 in the third term of (5.1) changes with a change in ω.

In view of the independence of δr_{da}, δr_{dc} and δr_{dk}, caused by the fact that the point *d* is a singular point, we obtain from (5.3) and (5.4) the conditions for transversality:

$$\frac{dF(r_0)}{dr_0} + \Phi_0 = 0, \qquad \frac{dF(r_4)}{dr_4} + \Phi_4 = 0, \qquad \Phi_4 - \Phi_0 = 0 \tag{5.5}$$

The second of these conditions is subject to the solution (3.6) of Euler's equations for the variational problem [4]. The first condition of (5.5), in which Φ_0 is substituted by Φ_4 by virtue of the third condition of (5.5), and the third condition in expanded form have the following forms, respectively

$$\tau(\alpha_0)\sigma(\alpha_0)\left[\frac{\sin\alpha_0}{\varkappa}+\frac{\cos\vartheta_0}{\sin(\vartheta_0+\alpha_0)}\right]-\tau(\alpha_4)\sigma(\alpha_4)\left[\frac{\sin\alpha_4}{\varkappa}-\frac{\cos\vartheta_4}{\sin(\vartheta_4-\alpha_4)}\right]+$$
$$+\Lambda\left[\cot(\vartheta_0+\alpha_0)-\cot(\vartheta_4-\alpha_4)\right]-\mu\left[\frac{\tau(\alpha_0)}{\sin(\vartheta_0+\alpha_0)}+\frac{\tau(\alpha_4)}{\sin(\vartheta_4-\alpha_4)}\right]=0 \quad (5.6)$$

$$\tau(\alpha_0)\sigma(\alpha_0)\left[\frac{\sin\alpha_0}{\varkappa}-\frac{\cos\vartheta_0}{\sin(\vartheta_0-\alpha_0)}\right]-\tau(\alpha_4)\sigma(\alpha_4)\left[\frac{\sin\alpha_4}{\varkappa}-\frac{\cos\vartheta_4}{\sin(\vartheta_4-\alpha_4)}\right]+$$
$$+\Lambda\left[\cot(\vartheta_0-\alpha_0)-\cot(\vartheta_4-\alpha_4)\right]+\mu\left[\frac{\tau(\alpha_0)}{\sin(\vartheta_0-\alpha_0)}-\frac{\tau(\alpha_4)}{\sin(\vartheta_4-\alpha_4)}\right]=0 \quad (5.7)$$

where

$$\Lambda=\tau(\alpha_4)\sigma(\alpha_4)\frac{\sin^2\vartheta_4}{\cos\alpha_4},\qquad \mu=\sigma(\alpha_4)\frac{\cos(\vartheta_4+\alpha_4)}{\cos\alpha_4} \quad (5.8)$$

The solution is permissible only when the inequalities (2.6) and (2.7) are satisfied in addition to the conditions (5.6) and (5.7).

We notice that the equations $\alpha_4=\alpha_0$, $\vartheta_4=\vartheta_0$ are double roots for the Equations (5.6) to (5.7). If α_0 and ϑ_0 are moreover connected by the equation $T(\alpha_0,\vartheta_0)=0$, where $T(\alpha,\vartheta)$ is defined in (4.2), then we have a treble root. We present examples of the solution of Equations (5.6) to (5.8) for certain values of α_0, ϑ_0:

α_0	ϑ_0	α_1	ϑ_1	α_4	ϑ_4
0.0500	—0.3000	0.0914	—0.0991	0.0887	—0.0860
0.1000	—0.5000	0.1711	—0.1807	0.1679	—0.1671
0.1664	—0.6686	0.2670	—0.2789	0.2643	—0.2693
0.2000	—0.9000	0.3663	—0.3475	0.3586	—0.3268

In these examples the points α_0, ϑ_0 and α_4, ϑ_4, in the α, ϑ plane lie on different sides of the curve VSU, corresponding to the equation $T(\alpha,\vartheta)=0$. In the solutions presented the inequalities (2.6) and (2.7) are satisfied.

6. The region of shockless solutions. In the original formulation of the problem the flow was assumed to be isentropic and the value of ϕ was assumed equal to unity. Let us now determine the region in which this assumption actually secures minimum χ.

We shall assume that the value of $\phi(r)$ is variable. The second principle of thermodynamics demands the limitation

$$\varphi(r)\geqslant 1 \quad (6.1)$$

The first variation δI when $\phi = \phi(r)$ has the form

$$\delta I = \left(\frac{dF(r_0)}{dr_0} + \Phi_4\right)\delta r_{da} + \left(\frac{dF(r_4)}{dr_4} + \Phi_4\right)\delta r_{dk} + (\Phi_4 - \Phi_0)\,\delta r_{dc} +$$

$$+ \nu_d\left[\left(\frac{d\beta}{dr}\right)_{dk} - \left(\frac{d\beta}{dr}\right)_{db}\right]\delta r_{d\bar{k}} + \nu_d\frac{\sin 2\alpha_d}{2\varkappa(\varkappa-1)\varphi_d}\left[\left(\frac{d\varphi}{dr}\right)_{dk} - \left(\frac{d\varphi}{dr}\right)_{db}\delta r_{dk} - \nu_b\left[\delta\beta_b +\right.\right.$$

$$\left. + \frac{\sin 2\alpha_b}{2\varkappa(\varkappa-1)\varphi_b}\delta\varphi_b\right] + \int\limits_{r=r_b}^{r_d}\left[\Phi_\alpha\delta\alpha + \left(\Phi_\beta - \frac{d}{dr}\Phi_{\beta'}\right)\delta\beta + (\Phi_\varphi) - \frac{d}{dr}\Phi_{\varphi'}\delta\varphi)\right]dr \qquad (6.2)$$

The multiplier ν is determined by equating to zero the expression with $\delta\beta$ under the integral sign

$$\frac{1}{\sin^2(\vartheta-\alpha)}\left\{r\tau(\alpha)\left[\sigma(\alpha)\cos\alpha - \mu\cos(\vartheta-\alpha)\right]\varphi^{-\frac{1}{\varkappa-1}} -\right.$$

$$\left. - \lambda + \frac{\nu}{r}\sin^2\alpha\right\} - \frac{d\nu}{dr} = 0 \qquad (6.3)$$

When we have fulfilled the conditions of transversality, Equation (6.3) and the equations $\delta X = \delta\Psi = 0$, specified in the formulation of the problem, the variation $\delta\chi$ has the form

$$\delta\chi = -\int\limits_{r=r_b}^{r_d}\left[\Phi_\alpha\delta\alpha + \left(\Phi_\varphi - \frac{d}{dr}\frac{\nu\sin 2\alpha}{2\varkappa(\varkappa-1)\varphi}\right)\delta\varphi\right]\sin(\vartheta-\alpha)\,dl \qquad (6.4)$$

where

$$\Phi_\alpha = \frac{r}{\sin^2(\vartheta-\alpha)}\left\{\sigma(\alpha)\tau(\alpha)\left[\frac{\varkappa+1}{2}\frac{\sin 2\vartheta}{\sin\alpha} - 2(1+\cos 2\alpha)\cos\alpha\right]\varphi^{-\frac{1}{\varkappa-1}} +\right.$$

$$+ \frac{\lambda}{r}(2\cos 2\alpha - \varkappa + 1) + \mu\tau(\alpha)\left[(2\cos 2\alpha - \varkappa + 1)\cos(\vartheta-\alpha) -\right.$$

$$\left. - (\varkappa+1)\cot\alpha\sin(\vartheta-\alpha)\right]\varphi^{-\frac{1}{\varkappa-1}} -$$

$$\left. - \frac{\nu}{2r^2}\left[\sin^2 2\alpha - 2(\varkappa - \cos 2\alpha)\sin^2\vartheta\right)\right]\right\}\frac{1}{\varkappa - \cos 2\alpha} \qquad (6.5)$$

$$\Phi_\varphi = \frac{r\tau(\alpha)}{\varkappa-1}\left\{\sigma(\alpha)\left[\frac{\sin\alpha}{\varkappa} - \frac{\cos\vartheta}{\sin(\vartheta-\alpha)}\right] + \frac{\mu}{\sin(\vartheta-\alpha)}\right\}\varphi^{-\frac{\varkappa}{\varkappa-1}} - \frac{\nu\sin 2\alpha\ \varphi'}{2\varkappa(\varkappa-1)\varphi^2} \qquad (6.6)$$

In the isentropic case $\nu = 0$, μ is determined by Equation (3.6), and the expression standing in the curly brackets of Formula (6.6), has the form

$$\frac{\sigma(\alpha)\tan}{\varkappa}\frac{(\cos^2\alpha - \varkappa)\sin\vartheta - \sin\alpha\cos\alpha\cos\vartheta}{\sin(\vartheta-\alpha)}$$

Accordingly, the function Φ_ϕ vanishes when

$$\vartheta = -\tan^{-1}\frac{\sin 2\alpha}{2\varkappa - 1 - \cos 2\alpha} + n\pi \qquad (6.7)$$

The relation (6.7) is portrayed in Fig. 5 by the curve VWU, intersecting the curve VSU at the points V, P, Q, U. It is easy to see that above the curve VWU the inequality $\Phi_\phi \sin(\vartheta - \alpha) < 0$ holds, whilst below the curve VWU the inequality $\Phi_\phi \sin(\vartheta - \alpha) > 0$ holds.

From (6.1) it follows that when $\phi = 1$ permissible variations $\delta\phi > 0$. On the basis of (6.4) we draw the following conclusion. In the region

$$-\tan^{-1}\frac{\sin 2\alpha}{2\varkappa - 1 - \cos 2\alpha} < \vartheta < \pi - \tan^{-1}\frac{\sin 2\alpha}{2\varkappa - 1 - \cos 2\alpha} \qquad (6.8)$$

permissible variations $\delta\phi$ lead to an increase in the resistance χ. Accordingly, in the region (6.8) minimal resistance is achieved with isentropic flows.

Let us consider the region Ω, bounded by the line VU, on which $\vartheta = 0$ (Fig. 5), the curve (4.3) on the portion VP, the curve (6.7) on the portion PWQ, and the curve (4.3) on the portion QU. Suppose that the starting point d (Fig. 8) of the extremal belongs to this region. In the region Ω we have $-\pi < \vartheta - \alpha < 0$, i.e. with motion along the extremal section of the characteristic from the point d to the point b the value of r decreases, or at least when $\vartheta = \alpha = 0$ it remains unchanged. This corresponds to motion along the extremal in the $\alpha\,\vartheta$ plane from the line VU to the portions $VPWQU$. Hence it follows that isentropic flows of the sort considered exist if the whole of the extremal belongs to the region Ω.

On extremals with $\vartheta > 0$ a similar limitation is not imposed even with sufficiently large ϑ, since the angle ϑ on the extremal portions of the characteristics does not change sign.

We notice that the form of the conditions of transversality, obtained from (6.2) for the region $PWQS$, differs from (5.6) to (5.7) in view of inequality (6.1).

With shockless flow in the region *adb* (Fig. 8) the shock wave *dn* does not exert any influence on the contour *ab*.

7. The order of calculation. Suppose we are given the points a and b and the characteristic ae. It is assumed that the solution of the problem relates to the type under consideration here.

First of all we work out the flow in the region *cad*. This can be done, for example, by the method of characteristics [12]. The point d in the region cad must be chosen so that the extremal characteristic db, obtained as a result of the calculation, comes to the point b. At a selected point d of the region cad the quantities a_0 and ϑ_0 are known. The values of a_4 and ϑ_4 are determined, satisfying Equations (5.6) to (5.8). It is then necessary to satisfy oneself that the values a_4 and ϑ_4 so determined belong to the region Ω, whilst the flow in the neighborhood of the point d corresponds to the type considered in Section 2. The latter is achieved by calculation of the flow in the neighborhood of the point d according to Formulas (2.1), (2.3) to (2.5) and verifying fulfillment of inequalities (2.7).

The values $a = a_4$ and $\vartheta = \vartheta_4$ so found, and also the known value of r at the point d, enable us to calculate λ and μ from Formulas (3.6). From Formulas (3.7) and (3.8) we construct the extremal db up to $r = r_b$. Correct choice of the coordinates of the point d is confirmed by fulfillment of the isoperimetric conditions (1.5).

The solution obtained must give minimum χ. This can be established at the minimum so obtained by verifying whether in fact all permissible variations lead to an increase in χ. Suppose, for example, that the characteristic cd is formed in the Ra plane by the curve cd, whilst the extremal db is the curve $d'b$ (Fig. 7). The condition $V < 0$ on the segment cd is verified by direct calculation according to Formulas (3.3) and (4.2). Permissible variations of a on cd are positive and lead to an increase in χ (boundary extremum). The segment $d'b$ gives rise to a two-sided minimum of χ, since above $d'b$ we have $V < 0$, whilst below $d'b$ we have $V > 0$. Formula (1.4) enables us to find the value of the wave resistance of the required body:

$$\sqrt{2(\varkappa + 1)}\pi \rho_0 \chi$$

To construct the profile ab of the body of revolution it is necessary to find x on the characteristic db from the formula

$$x = x_d + \int_{r=r_d}^{r} \cot (\vartheta - \alpha) dr \qquad (7.1)$$

Moreover the solution of Goursat's problem for Equation (1.1) with the known characteristic ad and a characteristic of the second kind, degenerating into the point d (the curve $d_0 d_1$ in Fig. 9), enables us to construct the flow in the region adk. By the solution of Goursat's problem with the known characteristic db and the characteristic of the first kind degenerating into the point d (the curve $d_1 d_4$ in Fig. 9), the flow in the region bdl is constructed. Finally, the solution of Goursat's problem for the characteristics kd and dl so obtained gives the flow in the region kdl. These calculations can also be carried out by the method of characteristics. The streamline ab found from the flow pattern is the required profile.

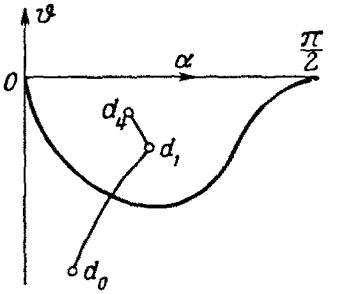

Fig. 9.

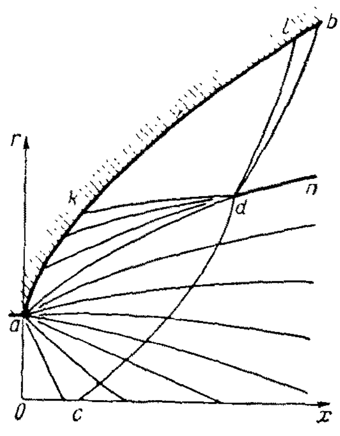

Fig. 10.

8. Optimal Laval nozzles. This name will be given to nozzles which, for a given flow across the inlet $a0$ (Fig. 10) and fixed points a and b, possess maximal thrust (minimum χ).

The solution of the problem of constructing an optimal nozzle can be obtained in a similar way to the solution of the problem just considered concerning the external flow past a body of revolution. Here we must remember that the characteristics of the first and second kinds have exchanged roles. Formally this reduces to saying that in the calculation Formulas (1.4), (1.5), (2.1), (2.3) to (2.5), (3.6) to (3.8), (5.6) to (5.8) and (7.1) the quantities a with various indices must be replaced by the quantities $-a$ with the same indices. The signs of the values of a in (2.7) must be conserved. The working diagrams for $\kappa = 1.4$ remain unchanged in Figs. 3-6, but in Figs. 5 and 6 the quantity ϑ must be replaced by $-\vartheta$. Further reasoning in this section is based on the assumption that all these substitutions in the formulas and figures mentioned have already been carried out.

It is necessary to notice the following peculiarity of flows in nozzles.

The region Ω corresponds to the inequality $\vartheta + a > 0$, showing that for motion along the extremal characteristic db from the point d to the point b the value of r increases (Fig. 10). Moreover in the Ra plane (Fig. 4) the motion originates in the side with the greatest R, whilst in the $\alpha\vartheta$ plane (Fig. 5) the motion is from the curve $VPWQU$ to the line VU. Consequently, if the point d belongs to the region Ω, then this point corresponds to the extremal db with arbitrarily large values of r_b.

If the surface of transition of an axisymmetric nozzle is plane and $\kappa = 1.4$, then the flow in the region of free expansion is bounded in the $\alpha\vartheta$ plane (Fig. 5) by the lines $\vartheta = 0$, $a = 0$ and the curve UT, representing a characteristic of the Prandtl-Meyer flow. From Fig. 5 it follows that the greater part of this region leads to isentropic extremal solutions and only a small sub-region in the neighborhood of the point P is connected with the formation of shock waves. The latter case occurs if the point d in the $\alpha\vartheta$ plane is sufficiently close to the curve UT, i.e. if the point b in the x, r plane is sufficiently close to the point a.

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